

THE MULTIPLICATION GROUP OF AN AG-GROUP

M. SHAH, A. ALI, I. AHMAD*, AND V. SORGE

ABSTRACT. We investigate the multiplication group of a special class of quasigroup called AG-group. We prove some interesting results such as: the multiplication group of an AG-group of order n is non-abelian group of order $2n$ and its left section is an abelian group of order n . The inner mapping group of an AG-group of any order is a cyclic group of order 2.

1. INTRODUCTION AND PRELIMINARIES

A groupoid G is an *AG-group* if it satisfies: (i) $(xy)z = (zy)x$, $\forall x, y, z \in G$. (ii) There exists left identity $e \in G$ (that is, $ex = x, \forall x \in G$). (iii) For every $x \in G$ there exists $x^{-1} \in G$ such that $x^{-1}x = xx^{-1} = e$. x and x^{-1} are called inverses of each other.

AG-group is a subclass of cancellative AG-groupoids [13]. Some basic properties of AG-groups have been derived in [11], and fuzzification of AG-groups can be seen in [12, 16]. AG-group is a generalization of abelian group and is a special quasigroup. AG-groups have been counted computationally in [18] and algebraically in [14]. The counting of AG-groups up to order 6 can also be found in [15]. AG-groups have been studied as a generalization of abelian group as well as a special case of quasigroups in [14]. The present paper discovers the multiplication group and inner mapping group of an AG-group. Multiplication group and inner mapping group of a loop have been investigated in a number of papers for example [1, 2, 3, 4, 5, 6, 7, 8, 9]. This has always been remained the most interesting topic of group theorists in loop theory. Quasigroup does not have inner mapping group because it does not have an identity element unless it is not a loop. But an AG-group though not a loop but it has a left identity so it has multiplication group as well as inner mapping group. We will prove here some interesting results about the multiplication group and inner mapping group of an AG-group

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* Corresponding Author.

that do not hold in case of a loop. For example for an AG-group G of order n the left section, L_S is an abelian group of order n . Its multiplication group is a nonabelian group of order $2n$. The inner mapping group of an AG-group is always a cyclic group of order 2 regardless of its order. The following lemma of [11] will be used in proofs .

Lemma 1. *Let G be an AG-group. Let $a, b, c, d \in G$ and e be the left identity in G . Then the following conditions hold in G .*

- (i) $(ab)(cd) = (ac)(bd)$ medial law;
- (ii) $ab = cd \Rightarrow ba = dc$;
- (iii) $a \cdot bc = b \cdot ac$;
- (iv) $(ab)(cd) = (db)(ca)$ paramedial law;
- (v) $(ab)(cd) = (dc)(ba)$;
- (vi) $ab = cd \Rightarrow d^{-1}b = ca^{-1}$;
- (vii) If e the right identity in G then it becomes left identity in G , i.e.,
 $ae = a \Rightarrow ea = a$;
- (viii) $ab = e \Rightarrow ba = e$;
- (ix) $(ab)^{-1} = a^{-1}b^{-1}$;
- (x) $a(b \cdot cd) = a(c \cdot bd) = b(a \cdot cd) = b(c \cdot ad) = c(a \cdot bd) = c(b \cdot ad)$;
- (xi) $a(bc \cdot d) = c(ba \cdot d)$;
- (xii) $(a \cdot bc)d = (a \cdot dc)b$;
- (xiii) $(ab \cdot c)d = a(bc \cdot d)$.

2. MULTIPLICATION GROUP OF AN AG-GROUP

Let G be an AG-group and a be an arbitrary element of G . The mapping $L_a : G \rightarrow G$ defined by $L_a(x) = ax$ is called left translation on G . Similarly the mapping $R_a : G \rightarrow G$ defined by $R_a(x) = xa$ is called right translation on G .

Our first result establish the relation between a left translation and a right translation.

Lemma 2. *Let G be an AG-group. Let $a, b \in G$ and e be the left identity in G . Then*

- (i) $L_a R_b = R_{ab}$.
- (ii) $R_a R_b = L_{ab}$.
- (iii) $L_a L_b = R_{(ae)} R_b$.
- (iv) $L_a L_b = L_{(ae)b} = L_{(be)a}$.
- (v) $R_a L_b = R_{(ae)b}$.
- (vi) $L_a L_b = L_b L_a$.
- (vii) $R_a L_b = R_b L_a$.

Proof. Let $a, b \in G$ and e be the left identity in G . Then

- (i) $L_a R_b(x) = L_a(xb) = a(xb) = x(ab) = R_{ab}(x) \Rightarrow L_a R_b = R_{ab}$.
- (ii) $R_a R_b(x) = R_a(xb) = (xb)a = (ab)x = L_{ab}(x) \Rightarrow R_a R_b = L_{ab}$.
- (iii) $L_a L_b(x) = L_a(bx) = a(bx) = (ea)(bx) = (xb)(ae) = R_{(ae)}(xb) = R_{(ae)} R_b(x) \Rightarrow L_a L_b = R_{(ae)} R_b$.
- (iv) By (ii) and (iii) and left invertive law.
- (v) $R_a L_b(x) = R_a(bx) = (bx)a = (bx)(ea) = (ae)(xb) = L_{ae}(xb) = L_{ae} R_b(x) \Rightarrow R_a L_b = L_{ae} R_b \Rightarrow R_a L_b = R_{(ae)b}$, by (i).
- (vi) $L_a L_b = L_{(be)a} \Rightarrow L_a L_b = L_b L_a$, by (iv).
- (vii) $R_a L_b = R_{(ae)b} = R_{(be)a} = R_b L_a$, by left invertive law and (v). ■

Remark 1. From Lemma 2 we note that if G is an AG-group, then the left translation L_a and the right translation R_a behave like an even permutation and an odd permutation respectively, that is;

$$L_a L_a = L_a, R_a R_a = L_a, L_a R_a = R_a, R_a L_a = R_a.$$

Next we recall the following definition.

Definition 1. Let G be an AG-group. Then the set $L_S = \{L_a : L_a(x) = ax \forall x \in G\}$ is called **left section** of G and the set $R_S = \{R_a : R_a(x) = xa \forall x \in G\}$ is called **right section** of G .

Definition 2. Let G be an AG-group. Then the set $\langle L_a, R_a : a \in G \rangle$ forms a group which is called **multiplication group of the AG-group G** and is denoted by $M(G)$ i.e $M(G) = \langle L_a, R_a : a \in G \rangle$.

We remark that left section of a loop is not a group but left section of an AG-group does form a group as we prove it in the following theorem.

Theorem 1. Let G be an AG-group of order n . Then L_S is an abelian group of order n .

Proof. By definition $L_S = \{L_a : L_a(x) = ax \forall x \in G, a \in G\}$. Let $L_a, L_b \in L_S$ for some $a, b \in G$. Then by Lemma 2 (iv), we have $L_a L_b = L_{(ae)b} \in L_S \Rightarrow L_S$ is an AG-groupoid. $L_e L_a = L_{(ee)a} = L_a$ and $L_a L_e = L_{(ae)e} = L_{(ee)a} = L_a$. Therefore L_e is the identity in L_S .

Let $L_a, L_b, L_c \in L_S$. Then $(L_a L_b) L_c = L_{(ae)b} L_c = L_{[(ae)b]e} = L_{(ce)((ae)b)} = L_{(ce)((be)a)} = L_{(ae)((be)c)} = L_a L_{(be)c} = L_a (L_b L_c)$.

Let $L_a \in L_S \Rightarrow a \in G \Rightarrow a^{-1} \in G \Rightarrow a^{-1}e \in G$. Let $a^{-1}e = b$ then $L_b \in L_S$. Now $L_a L_b = L_{(ae)b} = L_{(ae)(a^{-1}e)} = L_e = L_b L_a \Rightarrow L_b$ is the inverse of L_a .

Thus L_S is a group. Since from Lemma 2, we have $L_a L_b = L_b L_a$. Therefore L_S is an abelian group. ■

We illustrate the above result by an example.

Example 1. *An AG-group of order 3 :*

\cdot	0	1	2
0	0	1	2
1	2	0	1
2	1	2	0

The Multiplication group of the AG-group given in Example 1 is isomorphic to S_3 , the symmetric group of degree 3 as the following example shows.

Example 2. *Multiplication group of the AG-group given in Example 1.*

\cdot	L_0	L_1	L_2	R_0	R_1	R_2
L_0	L_0	L_1	L_2	R_0	R_1	R_2
L_1	L_1	L_2	L_0	R_2	R_0	R_1
L_2	L_2	L_0	L_1	R_1	R_2	R_0
R_0	R_0	R_1	R_2	L_0	L_1	L_2
R_1	R_1	R_2	R_0	L_2	L_0	L_1
R_2	R_2	R_0	R_1	L_1	L_2	L_0

Here $L_S = \{L_0, L_1, L_2\}$ which is an abelian group as the following table shows:

\cdot	L_0	L_1	L_2
L_0	L_0	L_1	L_2
L_1	L_1	L_2	L_0
L_2	L_2	L_0	L_1

But $R_S = \{R_0, R_1, R_2\}$ does not form an AG-group as the following table shows:

\cdot	R_0	R_1	R_2
R_0	L_0	L_1	L_2
R_1	L_2	L_0	L_1
R_2	L_1	L_2	L_0

Remark 1. *Right section does not form even an AG-groupoid.*

Lemma 2 guarantees that for an AG-group G , $M(G) = \langle L_a, R_a : a \in G \rangle = \{L_a, R_a : a \in G\}$.

Theorem 2. Let G be an AG-group of order n . The set $\{L_a, R_a : a \in G\}$ forms a non-abelian group of order $2n$ which is called multiplication group of the AG-group G and is denoted by $M(G)$ i.e $M(G) = \{L_a, R_a : a \in G\}$.

Proof. From Lemma 2, it is clear that $M(G)$ is closed. L_e plays the role of identity as $L_a L_e = L_e L_a = L_a$.

$$R_a L_e = R_{(ae)e} = R_{(ee)a} = R_a = R_{ea} = L_e R_a.$$

Let $L_a \in M(G) \Rightarrow a \in G \Rightarrow a^{-1} \in G \Rightarrow R_{a^{-1}} \in M(G)$ and $R_a R_{a^{-1}} = L_{aa^{-1}} = L_e = L_{a^{-1}a} = R_{a^{-1}} R_a$. Therefore $R_{a^{-1}}$ is the inverse of R_a in $M(G)$. Associativity in $M(G)$ follows from the associativity of mappings. Thus $M(G)$ is a group. Note that $M(G)$ is non-abelian because $R_a R_b \neq R_b R_a$ by 2 (ii). ■

To make things a bit more clearer we consider the following examples.

Example 3. An AG-group of order 4.

\cdot	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	3	2	1	0
3	2	3	0	1

Example 4. Multiplication group of the AG-group in Example 3.

\cdot	L_0	L_1	L_2	L_3	R_0	R_1	R_2	R_3
L_0	L_0	L_1	L_2	L_3	R_0	R_1	R_2	R_3
L_1	L_1	L_2	L_3	L_0	R_3	R_0	R_1	R_2
L_2	L_2	L_3	L_0	L_1	R_2	R_3	R_0	R_1
L_3	L_3	L_0	L_1	L_2	R_1	R_2	R_3	R_0
R_0	R_0	R_1	R_2	R_3	L_0	L_1	L_2	L_3
R_1	R_1	R_2	R_3	R_0	L_3	L_0	L_1	L_2
R_2	R_2	R_3	R_0	R_1	L_1	L_2	L_3	L_0
R_3	R_3	R_0	R_1	R_2	L_2	L_3	L_0	L_1

From Example 4 we have the following observations:

- (1) The multiplication group of an AG-group is not necessarily dihedral. For example, $(L_1 \cdot R_3)^2 = R_2^2 = L_3 \neq L_0$. So here $M(G)$ is not D_4 .
- (2) From Examples 2 and 4 the left sections in both the examples are C_3 and C_4 respectively.

Theorem 3. Let G be an AG-group. Let a be an element of G distinct from e . Then a is self-inverse $\iff R_a^{-1} = R_a$.

Proof. Suppose a is self-inverse. Since $R_a(x) = xa$, then R_a is of order 2, as $R_a(R_a(x)) = (xa)a = (xa)a^{-1} = x \implies R_a^2 = L_e \implies R_a^{-1} = R_a$.

Conversely let $R_a^2 = L_e$ then $R_a^2(x) = L_e(x) \forall x \in G \implies (xa)a = ex = x$. Now by left invertive law, $a^2x = x$. This by right cancellation implies $a^2 = e$ or $a^{-1} = a$. ■

Remark 2. R_a cannot fix all the elements of AG-group G . For if we suppose that R_a fixes all the elements. That is; $R_a(x) = x \forall x \in G \implies xa = x \forall x \in G \implies a$ is the right identity and hence G is abelian.

Theorem 4. For every AG-group G , the inner mapping group; $\text{Inn}(G) = \{L_0, R_0\}$ is isomorphic to C_2 .

Proof. As $R_a(0) = 0a = 0$. This implies that only R_0 maps 0 on 0. On the other hand $L_0(0) = 0$ and no other L_a can map 0 on 0. Because let $L_a(0) = 0$ where $a \neq 0$. Then $a0 = 0$. This implies $R_0(a) = 0$. But $R_0(0) = 0$. This implies that R_0 is not a permutation which is a contradiction. Hence $\text{Inn}(G) = \{L_0, R_0\} \cong C_2$. The following table verifies the claim.

\cdot	L_0	R_0
L_0	L_0	R_0
R_0	R_0	L_0

Hence the proof. ■

Again the following are some quick observations:

- (i) The $\text{Inn}(G)$ is not necessarily normal in $M(G)$ for example consider the multiplication group of the AG-group given in 3. Here $L_1 \{L_0, R_0\} = \{L_1, R_3\} \neq \{L_1, R_1\} = \{L_0, R_0\} L_1$.
- (ii) For every AG-group G , L_S being of index 2 is normal in $M(G)$ and hence $M(G)/L_S \cong C_2$.
- (iii) For every AG-group G , left multiplication group of G coincides with L_S and right multiplication group of G coincides with $M(G)$.

A non-associative quasigroup can be left distributive as well as right distributive but a non-associative AG-group can neither be left distributive nor right distributive as the following theorem shows.

Theorem 5. Every left distributive AG-group and every right distributive AG-group is abelian group.

Proof. Let G be a left distributive AG-group. Then $\forall a, b, c \in G$, we have

$$\begin{aligned} a(bc) &= (ab)(ac) \\ &= (aa)(bc), \text{ by Lemma 1 (i)} \\ \Rightarrow a &= aa, \text{ by right cancellation.} \end{aligned}$$

This further implies that G is an abelian group. The second part is similar. ■

A non-associative quasigroup can be left distributive as well as right distributive but a non-associative AG-group can neither be left distributive nor right distributive as the following theorem shows.

Theorem 6. If G is an AG-group then $M(G)$ cannot be the group of automorphisms of L .

Proof. Suppose on contrary that $M(G)$ is the group of automorphisms of G . It means that every element of $M(G)$ is an automorphism of G . Since $L_a, R_a \in M(G)$ for all $a \in G$. Thus L_a and R_a are both automorphisms of G . So we can write

$$\begin{aligned} (xy)L_a &= (x)L_a \cdot (y)L_a, \text{ since } L_a \text{ is homomorphism} \\ \Rightarrow a(xy) &= (ax)(ay) \text{ for all } x, y \in G \end{aligned}$$

Thus G is left distributive. Similarly,

$$\begin{aligned} (xy)R_a &= (x)R_a \cdot (y)R_a, \text{ since } R_a \text{ is homomorphism} \\ \Rightarrow (xy)a &= (xa)(ya) \text{ for all } x, y \in G \end{aligned}$$

Thus G is right distributive. Hence G is distributive, which is a contradiction to Theorem 5. Whence $M(G)$ of an AG-group G cannot be the group of automorphisms of G . ■

Theorem 7. Let e be the identity and x, y be any elements of an AG-group G . Then:

- (i) $R_x^{-1} = R_{x^{-1}};$
- (ii) $L_x^{-1} = L_{x^{-1}e}.$

Proof. (i) Since G satisfies the right inverse property. Therefore

$$\begin{aligned} (yx)x^{-1} &= y \\ \Rightarrow R_{x^{-1}}R_x(y) &= y = L_e(y) \forall x, y \in G \\ \Rightarrow R_{x^{-1}}R_x &= L_e \Rightarrow R_x^{-1} = R_{x^{-1}}. \end{aligned}$$

(ii) By Lemma 2 (iv)

$$\begin{aligned} L_xL_{x^{-1}e} &= L_{(xe)(x^{-1}e)} = L_{(xx^{-1})e} = L_e \\ \Rightarrow L_x^{-1} &= L_{x^{-1}e}. \end{aligned}$$

Hence the theorem. ■

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E-mail address: `shahmaths_problem@hotmail.com`

E-mail address: `dr_asif_ali@hotmail.com`

DEPARTMENT OF MATHEMATICS, QAUID-I-AZAM UNIVERSITY, ISLAMABAD, PAKISTAN.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MALAKAND, CHAKDARA DIR(L),
PAKISTAN.

E-mail address: `iahmaad@hotmail.com`

E-mail address: `V.Sorge@cs.bham.ac.uk`

SCHOOL OF COMPUTER SCIENCE, UNIVERSITY OF BIRMINGHAM, UK.